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Generalized Deformed Para-Bose Oscillator and Its Coherent States

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Generalized deformed commutation relations for a single-mode para-Bose oscillator are constructed. The connection of generalized deformed para-Bose oscillators with para-Bose oscillators is determined. From these, the energy spectrum of generalized deformed para-Bose oscillators and the coherent states of the annihilation operators are discussed.

In the last few years there has been increasing interest in particles obeying statistics different from Bose or Fermi statistics. These generalized statistics are called para-Bose and para-Fermi statistics (Green, 1953; Ohnuki and Kamefuchi, 1982; Biswas and Das, 1988; Greenberg, 1990). Since the advent of the theory of parastatistics there have been many attempts to generalize the canonical commutation relations (CR). In particular, quantum deformations of the Heisenberg algebra and their possible physical applications have been extensively investigated (Greenberg, 1990; Macfarlane, 1989; Biedenharn, 1989; Chaichian and Kulish, 1990; Chaturvedi and Srinivasan, 1991a,b; Krishna Kumari *et al.*, 1992; Gangopadhyay, 1991; Man'ko *et al.*, 1993; Dutta-Roy and Ghosh, 1993; Oh and Sing, 1993; Truong, 1994; Chaichian and Demichev, 1994; Vokos and Zachos, 1994; Bang, 1994, 1995; Ellinas and Sobczyk, 1994; Chakrabarti and Jagannathan, 1994; Daskaloyannis, 1991, 1992; Bonatsos and Daskaloyannis, 1992; McDermott and Salomon, 1994; Shanta *et al.*, 1994). Chaturvedi and Srinivasan (1991b) have shown that a single para-Bose oscillator may be regarded as a deformed Bose oscillator. Krishna Kumari *et al.* (1992) constructed the CR for a single mode

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of the harmonic oscillator which contains para-Bose and q-oscillator CR. The connection of q-deformed para-Bose oscillators with para-Bose oscillators has been determined in Bang (1994). Chakrabarti and Jagannathan (1994) have considered a two-parameter deformation of a parabosonic algebra underlying the two-particle Calogero model and have discussed corresponding coherent states. It is noteworthy that starting from an arbitrary deformation of the oscillator, Daskaloyannis (1991) has constructed all of the general deformed oscillator algebra and has studied its properties. Next, the generalized deformed oscillators corresponding to the energy spectrum of the Pöschl–Teller potential (Daskaloyannis, 1992) or satisfying the same CR as a system of fermion pairs of zero angular momentum in a single-j shell (Bonatsos and Daskaloyannis, 1992) were discussed. Furthermore, the analogs of the single-photon and multiphoton coherent states for the generalized deformed bosonic oscillator system have been constructed in a unified way (Shanta *et al.*, 1994). Taking into consideration the expressions of the quantum fields in terms of the generalized deformed oscillators, we have found some special deformations in which corresponding deformed quantum fields are local and deformed oscillators are linear (Bang, 1995).

The following questions can be raised: Is it possible to construct generalized deformed CR for a single-mode para-Bose oscillator? Is there a connection between generalized deformed para-Bose oscillators and para-Bose oscillators? The main purpose of this work is devoted to these questions. In addition, we discuss the energy spectrum and the coherent states corresponding to generalized deformed para-Bose oscillators.

As is well known, a single-mode para-Bose system is characterized by the CR (Chaturvedi and Srinivasan, 1991; Krisha Kumari, 1992)

$$[a, \mathcal{N}] = a, \qquad [a^+, \mathcal{N}] = -a^+ \tag{1}$$

where

$$\mathcal{N} = \frac{1}{2}(aa^+ + a^+a) - \frac{p}{2} \tag{2}$$

and p is the order of the para-Bose system.

Also,

$$aa^+ = \tilde{f}(\mathcal{N} + 1), \qquad a^+a = \tilde{f}(\mathcal{N}) \tag{3}$$

with

$$\tilde{f}(n) = n + \frac{1}{2}\{1 - (-1)^n\}(p - 1) \tag{4}$$

Hence

$$[a, a^+] = \tilde{f}(\mathcal{N} + 1) - \tilde{f}(\mathcal{N}) = 1 + (-1)^{\mathcal{N}}(p - 1). \tag{5}$$

From these relations, an operator A^+ is constructed so that (Chaturvedi and Srinivasan, 1991; Krishna Kumari *et al.*, 1992)

$$A^+ = a^+ \frac{\mathcal{N} + 1}{\bar{f}(\mathcal{N} + 1)} \tag{6}$$

$$[a, A^+] = 1, \qquad [A^+, \mathcal{N}] = -A^+ \tag{7}$$

By the results just above mentioned the number operator $\mathcal{N}$ can be written as follows:

$$\mathcal{N} = A^+ a \tag{8}$$

Let us now turn to the question of a generalized deformed para-Bose oscillator corresponding to the annihilation and creation operators $\tilde{a}$ and $\tilde{a}^+$, respectively. According to the method of Daskaloyannis (1991, 1992), we begin by establishing an operator $\tilde{A}^+$ and consider the generalized deformed CR

$$\tilde{a}\tilde{A}^+ = g(\tilde{A}^+\tilde{a}) \tag{9}$$

where $\tilde{A}^+$ is defined by (Chaturvedi and Srinivasan, 1991, Krishna Kumari *et al.*, 1992)

$$\tilde{A}^+ = \frac{\tilde{a}^+(N + 1)}{\bar{f}(N + 1)} \tag{10}$$

In the q-deformed oscillator algebra the function $g(x)$ takes the form

$$g(x) = 1 + qx \tag{11}$$

The number operator N satisfies, by definition, the CR

$$[\tilde{a}, N] = \tilde{a}, \qquad [\tilde{A}^+, N] = -\tilde{A}^+ \tag{12}$$

It can be shown that N is given by (Daskaloyannis, 1991, 1992)

$$N = f(\tilde{A}^+\tilde{a}) \tag{13}$$

where the function f and the inverse function $F = f^{-1}$ are related to the function g as follows (Daskaloyannis, 1991, 1992)

$$g(x) = F\{1 + f(x)\} \tag{14}$$

If $|n\rangle$ is a orthonormalized base of eigenvectors of the number operator N,

$$N|n\rangle = n|n\rangle \tag{15}$$

$$\langle n|m\rangle = \delta_{nm} \tag{16}$$

then from (16) one has

$$\tilde{a}|n\rangle = \left(\tilde{f}(n)\,\frac{[n]}{n}\right)^{1/2} |n-1\rangle \tag{17}$$

$$\tilde{A}^{+}|n\rangle = ([n+1])^{1/2}|n+1\rangle \tag{18a}$$

$$\tilde{a}^{+}|n\rangle = \left(\tilde{f}(n+1)\,\frac{[n+1]}{n+1}\right)^{1/2} |n+1\rangle \tag{18b}$$

where

$$|n\rangle = \frac{1}{\{\tilde{f}(n)!\;[n]!/n!\}^{1/2}}\,(\tilde{a}^{+})^{n}|0\rangle \tag{19}$$

and

$$[n]! = \prod_{k=1}^{n}[k] = \prod_{k=1}^{n}F(k), \qquad [0]! = 1, \qquad \tilde{f}(n)! = \prod_{k=1}^{n}\tilde{f}(k) \tag{20}$$

We observe, further, that

$$\tilde{a}^{+}\tilde{a} = \tilde{f}(N)\,\frac{[N]}{N}, \qquad \tilde{a}\tilde{a}^{+} = \tilde{f}(N+1)\,\frac{[N+1]}{N+1} \tag{21}$$

With the help of (21) we obtain

$$[\tilde{a}, \tilde{a}^{+}] = \tilde{f}(N+1)\,\frac{[N+1]}{N+1} - \tilde{f}(N)\,\frac{[N]}{N} \tag{22}$$

So, we have just constructed the most general CR (22) for generalized deformed para-Bose oscillators. It includes the various forms of the CR defined in the literature (Krishna Kumari *et al.*, 1992; Bang, 1994; Chakrabarti and Jagannathan, 1994) as special cases. For example, the q-deformed para-Bose CR (Krishna Kumari *et al.*, 1992; Bang, 1994) can be reproduced with the following choice of the structure functions:

$$F(x) = \frac{q^{x}-1}{q-1}, \qquad f(x) = \frac{\ln\{1+(q-1)x\}}{\ln q}$$

Let us now turn to the problem of the connection between generalized deformed para-Bose oscillators and para-Bose oscillators. The boson realization of the algebra (12), following from (15), (17), and (18a), is given by

$$\tilde{a} = a\left(\frac{[N]}{N}\right)^{1/2} \tag{23}$$

$$\tilde{A}^{+} = \left(\frac{[N]}{N}\right)^{1/2} A^{+} \tag{24}$$

$$N = A^{+}a = \mathcal{N} \tag{25}$$

Combining (6), (10), and (24) yields

$$\tilde{a}^{+} = \left(\frac{[N]}{N}\right)^{1/2} a^{+} \tag{26}$$

So, owing to relations (23) and (26), we have just expressed the operators $\tilde{a}$ and $\tilde{a}^{+}$ of a generalized deformed para-Bose oscillator in terms of the operators a and a^{+} of a single-mode para-Bose oscillator. This result is general and can be applied for any deformed para-Bose oscillators, including the q-deformed ones (Bang, 1994).

As an important immediate consequence of (26) we have

$$[\tilde{a}^{+}, N] = -\tilde{a}^{+} \tag{27}$$

This formula will be used later.

Now, by the above relations we can also obtain the energy spectrum corresponding to the energy operator $H = (\lambda/2)(\tilde{a}^{+}\tilde{a} + \tilde{a}\tilde{a}^{+} + p - 1)$. This energy is

$$E_n = \frac{\lambda}{2}\left\{\tilde{f}(n)\frac{[n]}{n} + \tilde{f}(n+1)\frac{[n+1]}{n+1} + p - 1\right\} \tag{28}$$

As $p \to 1$ this becomes

$$E_n = \frac{\lambda}{2}\{[n] + [n+1]\} \tag{29}$$

Therefore the expression (27) reduces to the energy spectrum of the generalized deformed ordinary oscillator (Daskaloyannis, 1991, 1992). Next, letting $[n] \to n$ in (28), we get also the energy spectrum of a para-Bose oscillator.

In the final part of this paper we consider the construction of coherent states of the annihilation $\tilde{a}$, that is,

$$\tilde{a}|z\rangle = z|z\rangle \tag{30}$$

where z is a complex number.
The construction of these coherent states is most easily done following a simple technique (Chaturvedi and Srinivasan, 1991a,b) applicable to any generalized boson oscillator. Let

$$C = \tilde{a}\frac{N}{[N]}, \qquad C^{+} = \frac{N}{[N]}\tilde{A}^{+} \tag{31}$$

such that

$$[\tilde{a}, C^{+}] = 1, \qquad [C, \tilde{a}^{+}] = 1, \qquad [N, C] = -C, \qquad [N, C^{+}] = C^{+} \quad (32)$$

The coherent states will have the form

$$|z\rangle \sim \exp(zC^{+})|0\rangle \tag{33}$$

By using (10), (27), and (31) finally the normalized coherent state is

$$|z\rangle = \left\{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{[n]!}\right\}^{-1/2} \exp\left\{z\frac{N^{2}}{[N]\tilde{f}(N)}\tilde{a}^{+}\right\}|0\rangle \tag{34}$$

In the special case of $p = 1$, one obtains an expression

$$|z\rangle = \left\{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{[n]!}\right\}^{-1/2} \exp\left\{z\frac{N}{[N]}\tilde{a}^{+}\right\}|0\rangle \tag{35}$$

known in the generalized bosonic oscillator literature (Shanta *et al.*, 1994). Therefore it also contains the coherent state of the annihilation operator of a q oscillator (Chaturvedi and Srinivasan, 1991a) in the $p = 1$, $[x] = (q^{x} - 1)/(q - 1)$ case. These coherent states (34) for the cases of q-deformed and (p,q)-deformed para-Bose oscillators coincide with the known relations of the coherent states of Krishna Kumari *et al.* (1992) and Chakrabarti and Jagannathan (1994). Besides, this state includes the various forms of the coherent states defined in McDermott and Salomon (1994). It has already been shown that our expression (34) extends and generalizes some of the corresponding ones in several special cases (Chaturvedi and Srinivasan, 1991a; Krishna Kumari *et al.*, 1992; Chakrabarti and Jagannathan, 1994; McDermott and Salomon, 1994; Shanta *et al.*, 1994).

We conclude this paper with the following comments. The eigenvalues of a generalized deformed para-Bose oscillator (28) are of great importance since we can use them for the calculation of the masses of several particles. Our results may also lead to constructing the analogs of the single-photon and multiphoton coherent states for generalized deformed parabosonic oscillator systems. Another interesting problem is the study of the solution of the general equation (28), which determines the structure function $F(x)$ from a given energy spectrum E_n, and deriving the equivalent deformed para oscillator. We will study these topics elsewhere.

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